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Conformal Anomaly and Large Scale Gravitational Coupling

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Abstract

We present a model in which the breackdown of conformal symmetry of a quantum stress-tensor due to the trace anomaly is related to a cosmological effect in a gravitational model. This is done by characterizing the traceless part of the quantum stress-tensor in terms of the stress-tensor of a conformal invariant classical scalar field. We introduce a conformal frame in which the anomalous trace is identified with a cosmological constant. In this conformal frame we establish the Einstein field equations by connecting the quantum stress-tensor with the large scale distribution of matter in the universe.

In the absence of a full theory of quantum gravity, one of the theoretical frameworks in which we may improve our understanding of quantum processes in a gravitational field is a semiclassical approximation. In this framework the matter is described by quantum field theory while the gravitational field itself is regarded as a classical object. The gravitational coupling of a quantum field is then investigated through the study of the quantum stress-tensor, i.e. the expectation value of the stress-tensor of the quantum field taken in some physical state. However, since the quantum stress-tensor contains singularities, some renormalization prescriptions [1] are used to obtain a meaningful expression. One of the most remarkable consequences of these prescriptions is the so called conformal anomaly. This means that the trace of the quantum stress-tensor of a conformal invariant field obtains a nonzero expression while the trace of the classical stress-tensor vanishes identically. The appearance of a nonvanishing trace may be regarded as the breackdown of conformal symmetry. Since the conformal invariance of a theory reflects its invariance under rescaling of lengths, one may expect that the anomalous trace of the quantum stress-tensor may somehow be related to a scale of length.

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The purpose of this note is to establish this relation by connecting the trace anomaly with a cosmological constant. We shall deal with this possibility by making a distinction between the trace anomaly and the traceless part of the quantum stress-tensor. Such a distinction is suggested by the results of the renormalization theory, although there is an alternative motivation which can be found in [2]. In this way a simple dynamical model is introduced in which the traceless part is characterized by the stress-tensor of a conformal invariant scalar field. In this model, it is possible to rescale the trace anomaly and convert it into a cosmological length. The conformal symmetry is then broken by this cosmological length and a preferred conformal frame is determined in which the Einstein field equations with a cosmological constant are established.

Let us begin with the results of renormalization of a quantum stress-tensor $\Sigma_{\alpha\beta}$ for a quantum scalar field conformally coupled with a background metric $g_{\alpha\beta}$[3] ‡

$$\nabla^{\alpha}\Sigma_{\alpha\beta} = 0 \tag{1}$$

$$\Sigma^{\alpha}_{\alpha} = -2v_1(x) \tag{2}$$

where

$$v_1(x) = \frac{1}{720}\{\Box R - R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\delta\gamma}R^{\alpha\beta\delta\gamma}\} \tag{3}$$

Here ∇_α denotes a covariant differentiation and $\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$; $R_{\alpha\beta\delta\gamma}$ is the Riemann curvature tensor, $R_{\alpha\beta}$ is the Ricci tensor and R is the curvature scalar. The first equation is a conservation law and the second one indicates an anomalous trace emerging from the renormalization process. The implication of Eq.(2) is that the quantum stress-tensor $\Sigma_{\alpha\beta}$ may be written in the following general form

$$\Sigma_{\alpha\beta} = \Sigma^{(0)}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}v_1(x) \tag{4}$$

where $\Sigma^{(0)}_{\alpha\beta}$ is a traceless tensor. In this way, $\Sigma_{\alpha\beta}$ is decomposed into two parts: a traceless part $\Sigma^{(0)}_{\alpha\beta}$ which respects the conformal symmetry and an anomalous part reflecting the quantum characteristics. The traceless condition of $\Sigma^{(0)}_{\alpha\beta}$ is automatically satisfied if we introduce a conformally invariant C-number scalar field ϕ satisfying

$$(\Box - \frac{1}{6}R)\phi = 0 \tag{5}$$

and identify $\Sigma^{(0)}_{\alpha\beta}$ with the stress-tensor of ϕ, namely[5]

$$T_{\alpha\beta}[\phi] = (\frac{2}{3}\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{6}g_{\alpha\beta}\nabla_\gamma\phi\nabla^\gamma\phi) - \frac{1}{3}(\phi\nabla_\alpha\nabla_\beta\phi - g_{\alpha\beta}\phi\Box\phi) + \frac{1}{6}\phi^2 G_{\alpha\beta} \tag{6}$$

in which $G_{\alpha\beta}$ is the Einstein tensor. The relation (4) takes then the form

$$\Sigma_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}v_1(x) \tag{7}$$

‡We use the units in which $\hbar = c = 1$ and follow the sign conventions of Hawking and Ellis [4].

The tracelessness of $T_{\alpha\beta}$ is then ensured by (5).
We may try to take the relation (7) as a general condition imposed on $\Sigma_{\alpha\beta}$ in the given fixed background geometry described by the metric tensor $g_{\alpha\beta}$. However, in this case due to the Eq.(1) and the nonvanishing trace anomaly, $T_{\alpha\beta}$ can not be expressed as a conserved stress-tensor and the anomalous trace would provide a quantum source for the traceless tensor $T_{\alpha\beta}$. Although such a source could be considered as a desirable dynamical characteristic in many contexts, but we should note that its appearance on the whole background metric stands in conflict with the dynamical equation (5) for ϕ, as a C-number field. For this reason we shall follow a different interpretation for the relation (7).
We first note that the conformal coupling of the scalar field ϕ implies that no distinction can be made among different conformally related configurations of ϕ and $g_{\alpha\beta}$ in terms of the dynamical equation (5). Thus, the question which presents itself is that which conformal frame should be taken as the physical frame. We shall choose the conformal frame by the condition that $T_{\alpha\beta}$ as the stress-tensor of the scalar field, ϕ, should actually be conserved. Following this strategy we consider a conformal transformation

$$\bar{g}_{\alpha\beta} = \Omega^2(x) g_{\alpha\beta}$$

$$\bar{\phi}(x) = \Omega^{-1}(x)\phi(x) \tag{8}$$

and write Eq.(7) in a conformal frame described by the metric $\bar{g}_{\alpha\beta}$ so that

$$\bar{\Sigma}_{\alpha\beta} = \bar{T}_{\alpha\beta} + \frac{1}{6}\Lambda \bar{g}_{\alpha\beta}\bar{\phi}^2 \tag{9}$$

or, equivalently

$$\bar{G}_{\alpha\beta} + \Lambda \bar{g}_{\alpha\beta} = 6\bar{\phi}^{-2}(\bar{\Sigma}_{\alpha\beta} + \tau_{\alpha\beta}(\bar{\phi})) \tag{10}$$

where Λ denotes a cosmological constant which is taken to be related to the anomalous trace by

$$-3\bar{\phi}^{-2}\bar{v}_1(x) = \Lambda \tag{11}$$

The coefficient $\bar{\phi}^{-2}$ is introduced to make the dimension of both sides consistent. The tensor $\tau_{\alpha\beta}(\bar{\phi})$ is equal to $\bar{T}_{\alpha\beta}$ without $G_{\alpha\beta}$-term and coincides up to a sign with the so called modified stress-tensor [6]. The relation (11) is a constraint on the conformal factor and singles out a specific conformal frame, which we call the cosmological frame, in which the anomalous trace is related to a cosmological constant. In the cosmological frame, Λ serves to characterize a distinguished cosmological length scale which breaks down the conformal symmetry of $\bar{T}_{\alpha\beta}$.
Let us now consider the trace of Eq.(9)

$$\bar{\Sigma}^{\alpha}_{\alpha} \sim \Lambda \bar{\phi}^2 \tag{12}$$

Remarkably, this relation permits us to estimate the background average value of $\bar{\phi}$, if we measure the trace of $\bar{\Sigma}_{\alpha\beta}$ in the cosmological frame in terms of the large scale distribution of matter[§], $\bar{\Sigma}^{\alpha}_{\alpha} \sim M/R_0^3$ where M and R_0 are the mass and the radius of the universe, respectively.

[§] This argument seems to be reasonable since in the cosmological frame the local fluctuations of the quantum stress-tensor, which are characterized by the trace anomaly, are replaced by a cosmological constant term.

Actually if we take into account the empirical fact that the radius of the universe, R_0, coincides with its Schwarzschild radius $2GM$, where G is the gravitational constant, the Eq.(12) reduces to

$$\bar{\phi}^{-2} \sim G \tag{13}$$

provided $\Lambda \sim R_0^{-2}$ holds, in agreement with observations. Substituting this result into the Eq.(10) leads to

$$\bar{G}_{\alpha\beta} + \Lambda \bar{g}_{\alpha\beta} \sim G\bar{\Sigma}_{\alpha\beta} \tag{14}$$

which establishes the usual features of the Einstein field equations with a cosmological constant. Thus the cosmological frame provides a preferred frame in which the background average value of $\bar{\phi}$ plays the role of the gravitational constant. This result attributes two physical characteristics to this frame. Firstly, the tensor $\bar{T}_{\alpha\beta}$ becomes compatible with the conservation property of a stress-tensor, and secondly the large scale gravitational coupling is described by the Einstein field equations.

References

[1] N. D. Birrell and P. C. W. Davies, Quantum fields in curved space (Cambridge University Press, Cambridge, England, 1982)

[2] H. Salehi, Inter. J. Theo. Phys. **37**, 1253 (1998)

[3] R. M. Wald, Commun. Math. Phys. **54**, 1 (1977)
R. M. Wald, Phys. Rev. D **17**, 1477 (1978)

[4] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, Cambridge, England, 1973)

[5] S. Deser, Ann. Phys. **59**, 248 (1970)

[6] L. Parker, Phys. Rev. D **7**, 976 (1973)